INTERMEDIATE CHRISTOFFEL-MINKOWSKI PROBLEMS FOR FIGURES OF REVOLUTION

BY
WILLIAM J. FIREY

ABSTRACT

Necessary and sufficient conditions are described on a p function ψ over the unit sphere in Euclidean n-space E^n in order for ψ to be the pth order, elementary symmetric function of the prncipal radii on the boundary of a sufficiently smooth convex body of revolution in E^n; here these radii are taken as functions of the outer unit normal direction on the bounding surface; p satisfies $1 \leqq p < n - 1$.

A convex body K in Euclidean n-space E^n, with sufficiently smooth boundary, has $n-1$ associated elementary symmetric functions of principal radii which are defined as follows. Let $x(u)$ denote the unique boundary point of K at which the outer unit normal vector is u and let $R_1(u), \cdots, R_{n-1}(u)$ signify the principal radii of curvature of the boundary of K at $x(u)$. Set

$$F_p(u) = \sum R_{i_1}(u) \cdots R_{i_p}(u), \tag{1}$$

where the sum is extended over all increasing sequences $i_1, \cdots, i_p$ of indices chosen from the set $1, \cdots, n-1$. We may view F_p as a function over the unit sphere Ω of points u.

In this note we study a special case of the following problem: what are necessary and sufficient conditions for a function ψ, defined over Ω, to be a function F_p as described by (1), for preassigned p satisfying $1 \leqq p \leqq n-1$? We call this a Christoffel-Minkowski problem; the extreme cases $p = 1$ and $p = n-1$ were first studied by E. Christoffel and H. Minkowski. Complete solutions of Minkowski's problem were given by W. Fenchel and B. Jessen [8] and A. D.

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Aleksandrov [1]; those for Christoffel's problem are found in C. Berg [2] and W. Firey [6, 7]. The intermediate problems, where $1 < p < n-1$ and $n \geqq 4$ are almost completely open.

The work presented here is little more than an extension of results in a paper of Z. Nádeník [10] which treated Christoffel's problem for figures of revolution in the case $n = 3$. We derive necessary and sufficient conditions on a function ψ, defined over Ω, in order that there is a convex body of revolution K for which

$$\psi(u) = F_p(u)$$

for a preassigned p satisfying $1 \leqq p < n-1$. A few consequences of this result are also mentioned. It is hoped that the special cases examined here may suggest the nature of the complete solution for problems of this type.

The uniqueness question for Christoffel-Minkowski problems has long been settled: If K and K' have equal pth elementary symmetric functions of principal radii, then K and K' differ at most by a translation. For details see Buseman [4, p. 70].

1. Let K be a convex body of revolution in E^n, $n \geqq 3$. For convenience, we suppose Cartesian coordinates $x_1, \cdots, x_n$ introduced in such a way that the x_n-axis is the axis of revolution for K and the origin is an interior point of K; consequently those hyperplanes $x_n =$ const. which intersects K do so in $(n-1)$-dimensional balls. A meridian section of K is the intersection of K with a half-plane bounded by the x_n-axis. Such a section is a two-dimensional convex body. That part of the boundary of a meridian section which lies on the boundary of K itself is called a meridian of K. The intersection of the boundary of K with the x_n-axis consists of two points: the north pole of K lying on the positive half of the x_n-axis, and the south pole on the other half. These poles are common end points for all meridians. The intersection of K with any of its supporting hyperplanes is contained in a single meridian and the outer unit normal u to the hyperplane lies in the half-plane of that meridian. Clearly the distance from the origin to such a hyperplane depends only on the angle θ between u and the hyperplane $x_n = 0$. Here θ satisfies $-\pi/2 \leqq \theta \leqq \pi/2$. We write $h(\theta)$ for this distance.

The smoothness assumptions about K will be these: h has a continuous second derivative h'' and $h + h''$ is strictly positive over $-\pi/2 \leqq \theta \leqq \pi/2$. As a consequence, K is strictly convex and the mapping of the interval $-\pi/2 \leqq \theta \leqq \pi/2$ into any one meridian is topological. With respect to behaviour at the poles we must have $h'(\pm \pi/2) = 0$. It follows that

$$\lim_{\theta \to \pm\pi/2} h'(\theta)/\cos\theta = \mp h''(\pm \pi/2).$$

In view of the foregoing remarks, supporting hyperplanes are tangent hyperplanes and each meets K in a single point. We write $x(u)$ for that boundary point of K at which the tangent hyperplane has outer unit normal u. Suppose $x(u)$ is neither one of the poles of K: at $x(u)$ we have $n-2$ equal principal radii of curvature

$$R_2(u) = \cdots = R_{n-1}(u) = R(u)$$

whose common value is the distance of $x(u)$ from the x_n-axis, divided by $\sqrt{(1 - u_n^2)}$. The remaining principal radius $R_1(u)$ is the radius of curvature of the meridian through $x(u)$ at that point. In terms of h these radii are

$$R(u) = h(\theta) - h'(\theta)\tan\theta,\ R_1(u) = h(\theta) + h''(\theta). \tag{2}$$

The pth elementary symmetric function of the principal radii of K at $x(u)$ is given by

$$F_p(u) = \binom{n-2}{p} R^p(u) + \binom{n-2}{p-1} R_1(u) R^{p-1}(u). \tag{3}$$

Here F_p is a function of the latitude θ alone: we write

$$\phi_p(\theta) = F_p(u).$$

To put the connection between ϕ_p and h in a convenient form, we introduce the auxiliary function f defined by

$$f(\theta) = (h(\theta)\cos\theta - h'(\theta)\sin\theta)^p. \tag{4}$$

The functions f and f' given by

$$f'(\theta) = -p\sin\theta(h(\theta) + h''(\theta))(h(\theta)\cos\theta - h'(\theta)\sin\theta)^{p-1}, \tag{5}$$

are continuous over $-\pi/2 \leqq \theta \leqq \pi/2$. Since $f(\theta)$ is the pth power of the distance of $x(u)$ from the x_n-axis, it must be positive except at the poles of K.

The fundamental relation between f and ϕ_p is found from (2), (3), (4), (5) to be

$$f'(\theta) - (n-p-1)f(\theta)\tan\theta = -p\phi_p(\theta)\cos^{p-1}\theta\sin\theta \Big/ \binom{n-2}{p-1}. \tag{6}$$

As subsidiary conditions we have

$$\lim_{\theta \to \pm\pi/2} f(\theta)/\cos^p\theta \text{ exists} \tag{7}$$

by virtue of the behaviour of h at the poles of K. Consequently we have from (6) and the properties of f, as a first condition on ϕ_p: ϕ_p is continuous over $-\pi/2 < \theta < \pi/2$ and tends to finite limits as θ tends to $\pm\pi/2$.

Any solution of (6) has the representation

$$f(\theta) = \left(C - p\int_0^\theta \phi_p(\tau)\cos^{n-2}\tau\sin\tau\,d\tau\right) \Big/ \binom{n-2}{p-1}\cos^{n-p-1}\theta$$

for some choice of the constant C. In order for (7) to hold for $\theta \to = \pi/2$, C must be chosen so that

$$f(\theta) = p\int_\theta^{\pi/2} \phi_p(\tau)\cos^{n-2}\tau\sin\tau\,d\tau \Big/ \binom{n-2}{p-1}\cos^{n-p-1}\theta \tag{8}$$

for $-\pi/2 < \theta < \pi/2$. An application of L'Hospital's rule shows that f as given by (8) does satisfy (7) at the north pole of K.

The required positivity of f yields our next condition on ϕ_p:

$$\int_\theta^{\pi/2} \phi_p(\tau)\cos^{n-2}\tau\sin\tau\,d\tau > 0 \quad \text{for } -\pi/2 < \theta < \pi/2. \tag{9}$$

Further, the behaviour of f at the south pole of K shows that ϕ_p must satisfy

$$\int_{-\pi/2}^{\pi/2} \phi_p(\tau)\cos^{n-2}\tau\sin\tau\,d\tau = 0. \tag{10}$$

Under this assumption, L'Hospital's rule shows that (7) is also satisfied at the south pole.

The strict convexity of K entails the positivity of $R(u)$ and $R_1(u)$ as given by (2); the first of these conditions is taken care of by (9), the second yields a final requirement on ϕ_p. Because of the positivity of f and (5), we must have $f'(\theta)/\sin\theta < 0$: from (6) and representation (8) this gives

$$\phi_p(\theta) > (n-p-1)\int_\theta^{\pi/2} \phi_p(\tau)\cos^{n-2}\tau\sin\tau\,d\tau/\cos^{n-1}\theta. \tag{11}$$

for $-\pi/2 < \theta < \pi/2$.

So far we have proved the necessity part of the following assertion.

THEOREM. *In order for a function ψ over Ω to be the pth elementary symmetric function of the principal radii of a strictly convex body in Euclidean n-space which is a figure of revolution, it is necessary and sufficient that, in some system of geographic coordinates on Ω, ψ is a function ϕ of the latitude θ alone and, over $-\pi/2 < \theta < \pi/2$:*

(a) *ϕ is continuous and has finite limits as θ tends to $\pm\pi/2$,*

(b) $\int_\theta^{\pi/2}\phi(\tau)\cos^{n-2}\tau\sin\tau\,d\tau > 0$ *and is zero for* $\theta = -\pi/2$,

(c) $\phi(\theta) > (n-p-1)\int_\theta^{\pi/2}\phi(\tau)\cos^{n-2}\tau\sin\tau\,d\tau/\cos^{n-1}\theta$,

where $n \geqq 3,\ 1 \leqq p < n-1$.

2. For the sufficiency part, we need only observe that conditions (a) and (b) ensure the existence of a unique positive solution f to (6) which satisfies (7). With this f, as given by (8), we may now construct h from (4); (c) guarantees that $h + h''$ is positive. From all these facts the existence of a strictly convex body of revolution, with $\psi(u)$ as the pth elementary symmetric function of principal radii follows.

A few comments on these conditions are in order. Conditions (b) and (c) imply that ϕ is positive. Of course the positivity of ϕ does not ensure the satisfaction of (b) and (c); this observation is the basis of counter-examples to earlier, incomplete treatments of Christoffel's problem as well as intermediate Christoffel-Minkowski problems, see Aleksandrov [1] and Nádeník [10].

The latter part of condition (b), that is equation (10), is what is sometimes called the closure condition in Christoffel-Minkowski problems; for sufficiently smooth convex bodies, not necessarily figures of rotation, $F_p(u)$ must satisfy the vector equation

$$\int_\Omega uF_p(u)d\omega(u) = 0, \tag{12}$$

where $d\omega(u)$ is the area element of Ω at u, see Fenchel and Jessen [8]. In terms of $\phi_p(\theta)$, since

$$d\omega(u) = \cos^{n-2}\theta\,d\theta\,d\bar{\omega}(u),$$

where the differential form $d\bar{\omega}$ does not depend on θ, the last component equation in (12) reduces to (10). The remaining component equations of (12) hold by symmetry considerations alone.

3. We close with three simple consequences of the theorem.

If ψ satisfies (a), (b), (c) for some $p_0 < n-1$, then it does so also for all larger $p < n-1$; as a matter of fact this is even true for $p \leqq n-1$. Geometrically, if ψ is the p_0th elementary symmetric function of the principal radii of some convex body, then it is the pth elementary symmetric function of the principal radii of $n-p_0-1$ other convex bodies. Here the bodies in question must all be figures of revolution, but we conjecture that this is true generally.

If ψ_0 and ψ_1 satisfy (a), (b), (c) for the same p and the same system of geographic coordinates on Ω, and if α_0 and α_1 are non-negative numbers, not both zero, then ψ, defined by

$$\psi(u) = \alpha_0\psi_0(u) + \alpha_1\psi_1(u), \tag{13}$$

also satisfies (a), (b), (c). By this means new processes of combinations for coaxial convex bodies of revolution are defined, one for each p satisfying $1 \leqq p \leqq n-1$. Again, we conjecture that this can be done generally: it is certainly the case for $p = 1$, which corresponds to vector addition, as well as the case $p = n-1$, which corresponds to Blaschke addition. In this connection, see [5] and [9].

In (12) choose $\alpha_0 = 1-t$, $\alpha_1 = t$, $0 \leqq t \leqq 1$, and denote by K_t that convex body of revolution which has ψ as its pth elementary symmetric function of principal radii. As we noted in the introductory remarks, K_t is unique to within a translation. Two particular mixed volumes of K_t with the unit ball B, so-called Quermassintegrals, exhibit easily established and simple behaviour as functions of t:

$$V(K_t, p;\, B, n-p) = W_{n-p}(K_t),\quad V(K_t, p+1; B, n-p-1) = W_{n-p-1}(K_t).$$

Here, as a matter of notation, we write $V(C_1, n_1; \cdots; C_q, n_q)$ for the mixed volume of convex bodies $C_1, \cdots, C_q$ in E^n, where C_j is taken n_j times and $n_1 + \cdots + n_q = n$. The function $W_{n-p}(K_t)$ is linear in $1-t$ and t since

$$W_{n-p}(K_t) = \frac{1}{n}\int_\Omega \psi(u)d\omega(u) = (1-t)W_{n-p}(K_0) + tW_{n-p}(K_1). \tag{14}$$

This, and the succeeding integral formulas for Quermassintegrals are to be found in [3, p. 63]. If we write H_t for the support function of K_t, then

$$\begin{aligned} W_{n-p-1}(K_t) &= \frac{1}{n}\int_\Omega H_t(u)\psi(u)d\omega(u) \\ &= \frac{(1-t)}{n}\int_\Omega H_t(\omega)\psi_0(u)d\omega(u) + \frac{t}{n}\int_\Omega H_t(u)\psi_1(u)d\omega(\omega) \\ &= (1-t)V(K_t, 1; K_0, p; B, n-p-1) + tV(K_t, 1; K_1, p; B, n-p-1). \end{aligned}$$

To the right hand side we apply the inequalities of Fenchel and Aleksandrov, see Buseman [4, p. 50] in the form

$$V(K_t, 1; K_i, p; B, n-p-1) \geqq W_{n-p-1}^{1/(p+1)}(K_t)W_{n-p-1}^{p/(p+1)}(K_i),$$

for $i = 0, 1$. This yields

$$W_{n-p-1}^{p/(p+1)}(K_t) \geqq (1-t)W_{n-p-1}^{p/(p+1)}(K_0) + tW_{n-p-1}^{p/(p+1)}(K_1), \tag{15}$$

with equality if and only if K_0 and K_1 are homothetic. This last remark is a consequence of the conditions for equality in the Fenchel and Aleksandrov inequalities.

It should be kept in mind that (14) and (15) are proved here only for sufficiently smooth convex bodies of revolution which have parallel axes of revolution. Of course the general case will be true if our second conjecture turns out to be valid.

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Michigan State University, East Lansing
and
Oregon State University, Corvallis